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PATH INTEGRALS AND SPACETIME SYMMETRIES

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Applications of group theoretical methods in the path integral formulation of the quantum propagator are considered. The local spacetime symmetry of the Lagrangian is utilized for the path integration. Two examples are discussed explicitly.

1 INTRODUCTION

Since Feynman's *space-time approach to non-relativistic quantum mechanics* the path integral method has become a very important technique in quantum theories [1]. In this talk we present a method for the explicit path integral treatment based on the spacetime symmetry of a given problem. The local spacetime symmetry of the Lagrangian leads to an expansion of the short-time propagator in matrix elements of unitary irreducible representations of the space symmetry group. The path integration is performed using the orthogonality of the representations. Two examples are considered. Firstly, we discuss the free propagation of a non-relativistic particle in an n-dimesional space of constant negative curvature. Secondly, we will perform the path integral for a spinless relativistic particle in $(n+1)$-spacetime dimensions discribed by the Hamiltonian $H = c\sqrt{\mathbf{p}^2 + m^2c^2}$. Both exact results are compared with the semiclassical approximation.

2 GENERAL FORMULATION

According to Feynman the propagator of a quantum system may be obtained as a path integral [1]. In n-dimensional flat space (Euclidean or pseudo-Euclidean) the path integral is usually given in a sliced time basis

$$K(r,T) = \lim_{N\to\infty} \int \prod_{j=1}^{N-1} d\mathbf{x}_j \prod_{j=1}^{N} d\mathbf{p}_j \, (2\pi\hbar)^{-n} \times \exp\left\{(i/\hbar)(\mathbf{p}_j\Delta\mathbf{x}_j - \varepsilon H(\mathbf{p}_j,\mathbf{x}_j))\right\}. \quad (2.1)$$

Here r stands for the geodesic distance the system has propagated during the time T from $\mathbf{x}_i$ to $\mathbf{x}_f$. The notation is $t_j = t_0 + j\varepsilon$, $\varepsilon = T/N$, $\mathbf{x}_j = \mathbf{x}(t_j)$, $\Delta\mathbf{x}_j = \mathbf{x}_j - \mathbf{x}_{j-1}$, $\mathbf{x}_i = \mathbf{x}(t_0)$, $\mathbf{x}_f = \mathbf{x}(t_N)$ and similarly for $\mathbf{p}$. Performing the p-integration we find

$$K(r,T) = \lim_{N\to\infty} \int \prod_{j=1}^{N-1} d\mathbf{x}_j \prod_{j=1}^{N} K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon). \quad (2.2)$$

The explicit form of the short-time propagator depends on the Hamiltonian and is given, e.g., for $H = \mathbf{p}^2/2m + V(\mathbf{x})$ by

$$K(\mathbf{x}_j,\mathbf{x}_{j-1},\varepsilon) = (m/2\pi i\hbar\varepsilon)^{n/2} \times \exp\left\{\frac{i}{\hbar}\left(\frac{m}{2\varepsilon}(\Delta\mathbf{x}_j)^2 - V(\mathbf{x}_j)\varepsilon\right)\right\} \quad (2.3)$$

and for $H = c\sqrt{\mathbf{p}^2 + m^2c^2}$ by [2]

$$K(\mathbf{x}_j,\mathbf{x}_{j-1},\varepsilon) = 2ic\varepsilon\left(\frac{mc}{2\pi i\hbar}\right)^{\frac{n+1}{2}} (c^2\varepsilon^2 - (\Delta\mathbf{x}_j)^2)^{-(n+1)/4} \times K_{(n+1)/2}\left((imc/\hbar)\sqrt{c^2\varepsilon^2 - (\Delta\mathbf{x}_j)^2}\right). \quad (2.4)$$

Let G be a transformation group acting on the space $\mathcal{M}$ as $\mathbf{x} = \mathcal{D}^{(n)}(g)\,\mathbf{a}$, $g \in G$, $\mathbf{x},\mathbf{a} \in \mathcal{M}$. $\mathcal{D}^{(n)}(g)$ is an n-dimensional representation of G in $\mathcal{M}$. Now we choose a fixed vector $\mathbf{a}$ in $\mathcal{M}$. Any point $\mathbf{x}_j$ may be obtained from $\mathbf{a}$ through a transformation $\mathbf{x}_j = \mathcal{D}^{(n)}(g_j)\,\mathbf{a}$. Identifying the coordinates of $\mathbf{x}_j$ with group parameters of g_j we change the volume element $d\mathbf{x}_j$ on $\mathcal{M}$ into the normalized Haar measure dg_j of G, $d\mathbf{x}_j = |\mathcal{M}|dg_j$, where $|\mathcal{M}|$ stands for the volume of $\mathcal{M}$ for compact groups. If G is non-compact $|\mathcal{M}|$ follows from the identity $\int_{\mathcal{M}} d\mathbf{x} = |\mathcal{M}| \int_G dg$. Note that this is a formal equation. In general $\mathcal{M}$ may be viewed as a group quotient $\mathcal{M} = G/H$ where H is the stability group of $\mathbf{a}$. Therefore, a multiplication by $1 = \int_H dh_j$ is implied.

According to the transformation, the short-time propagator may be considered as a function over the group manifold of G. The propagation from $\mathbf{x}_{j-1}$ to $\mathbf{x}_j$ corresponds to a transformation by $g_j g_{j-1}^{-1}$ in the group manifold. We set $K(\mathbf{x}_j,\mathbf{x}_{j-1},\varepsilon) = K(g_{j-1}^{-1}g_j,\varepsilon)$,

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which can be expanded into a complete set $\{\ell\}$ of unitary irreducible representations in Hilbert space. Note that we may multiply each g_j by an arbitrary element of the stationary group H from the right, $\mathbf{x}_j = \mathcal{D}^{(n)}(g_j h_j)\,\mathbf{a} = \mathcal{D}^{(n)}(g_j)\,\mathbf{a}$. Obviously the short-time propagator is (even locally) invariant with respect to left and right multiplication of the subgroup H. Choosing a basis with $\langle \mathbf{a}|\mathcal{D}^{\ell}(g)|\mathbf{a}\rangle = \mathcal{D}^{\ell}_{00}(g)$, the Fourier decomposition of the short time action reduces to an expansion in zonal spherical functions [3]:

$$\begin{aligned} K(g_{j-1}^{-1}g_j,\varepsilon) &= \sum\!\!\!\!\!\!\int_{\ell} d_\ell\, f_\ell(\varepsilon)\, \mathcal{D}^{\ell}_{00}(g_{j-1}^{-1}g_j), \\ f_\ell(\varepsilon) &= \int_G K(g,\varepsilon)\, \mathcal{D}^{\ell\,*}_{00}(g)\, dg. \end{aligned} \tag{2.5}$$

The path integration may now be performed by using the orthogonality

$$\int_G \mathcal{D}^{\ell}_{00}(g_{j-1}^{-1}g_j)\, \mathcal{D}^{\ell'}_{00}(g_j^{-1}g_{j+1})\, dg_j = \frac{\delta(\ell,\ell')}{d_\ell}\, \mathcal{D}^{\ell}_{00}(g_{j-1}^{-1}g_{j+1}), \tag{2.6}$$

where $\delta(\ell,\ell') = \delta_{\ell\ell'}$ for discrete and $\delta(\ell-\ell')$ for continuous ℓ. Explicitly we have

$$\begin{aligned} K(r,T) &= \sum\!\!\!\!\!\!\int_{\ell} \exp\{-(i/\hbar)E_\ell T\}\,(d_\ell/|\mathcal{M}|)\mathcal{D}^{\ell}_{00}(g_0^{-1}g_N) \\ &= \sum\!\!\!\!\!\!\int_{\ell} \exp\{-(i/\hbar)E_\ell T\} \sum_{m=1}^{d_\ell} \Psi_{\ell m}(\mathbf{x}_f)\Psi^*_{\ell m}(\mathbf{x}_i), \end{aligned} \tag{2.7}$$

where

$$E_\ell = i\hbar|\mathcal{M}|f'_\ell(0), \;\; \Psi_{\ell m}(\mathbf{x}) = \sqrt{d_\ell/|\mathcal{M}|}\,\mathcal{D}^{\ell}_{m0}(g) \tag{2.8}$$

are the energy spectrum and normalized wave functions, respectively.

3 EXAMPLES

In the following we will discuss two examples within the above formalism. Firstly, we will consider the motion on a negatively curved space in n dimensions. This space may be viewed as a connected subspace of the group quotient $SO(n,1)/SO(n)$. Similar as $S^n = SO(n+1)/SO(n)$ is a space of constant positive curvature. Secondly, we will perform the path integral for the Klein-Gordon propagator in n space dimensions. The transformation group of R^n is the Euclidean group in n dimensions $E^n = T^n \otimes SO(n)$ and $R^n = E^n/SO(n) \simeq T^n$.

3.1 The Free Particle in a Space of Constant Negative Curvature

The line element ds of a uniform curved space with negative curvature $K = -1/R^2$ is

$$ds^2 = \left(1 + r^2/R^2\right)^{-1} dr^2 + r^2 d\vec{\omega}^2. \tag{3.1}$$

This geometry can be embedded in an $(n+1)$-dim. flat Minkowski space by setting [4]

$$x^o = R\cosh\chi,\;\; \mathbf{x} = R\sinh\chi\,\vec{\omega},\;\; ds^2 = -dx^{o\,2} + d\mathbf{x}^2, \tag{3.2}$$

where $\sinh\chi = r/R$, $\chi \in [0,\infty)$. We may identify the embedded surface with a "time-like" hyperboloid in $(n+1)$-dim. Minkowski space. Therefore,the Feynman ansatz for the propagator on a negative curved space reads [3,5]

$$\begin{aligned} K(r,T) &= \frac{1}{R^n}\lim_{N\to\infty}\int \prod_{j=1}^{N} K(g_{j-1}^{-1}g_j,\varepsilon)\prod_{j=1}^{N-1}|\mathcal{M}|dg_j, \\ K(g_{j-1}^{-1}g_j,\varepsilon) &= \left(mR^2/2\pi i\hbar\varepsilon\right)^{n/2} \\ &\quad\times \exp\left\{-(imR^2/\hbar\varepsilon)\left(1+\mathcal{D}^{(n+1)}_{00}(g_{j-1}^{-1}g_j)\right)\right\}, \end{aligned} \tag{3.3}$$

with $|\mathcal{M}| = 2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$. For the Fourier decomposition of the short-time propagator only the continuous fundamental series $\ell = -\frac{n-1}{2} + i\rho$, $\rho \geq 0$ of $SO(n,1)$ does contribute [3], where

$$d_\ell = 2|\Gamma((n-1)/2 + i\rho)|^2/[|\Gamma(i\rho)|^2\Gamma(n)]$$

and

$$f_\ell(\varepsilon) = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}}\left(\frac{2mR^2}{\pi i\hbar\varepsilon}\right)^{1/2}\exp\left\{-\frac{imR^2}{\hbar\varepsilon}\right\}K_{i\rho}\left(\frac{mR^2}{i\hbar\varepsilon}\right). \tag{3.4}$$

Note that $f'_\ell(0) = (\rho^2 + 1/4)\hbar/2imR^2$. The energy spectrum is, after fixing the energy scale to $E_0 = 0$, $E_\rho = \hbar^2\rho^2/2mR^2$. Using the explicit form of the zonal spherical functions in terms of Gegenbauer functions one may obtain the energy-dependent Green function [the Fourier transform of (3.3)] in closed form after lengthy calculation [6]:

$$\begin{aligned} G(r,E) &= (me^{2\pi i\epsilon}/\pi\hbar^2)\left(-2\pi R^2\sinh(r/R)\right)^{(2-n)/2} \\ &\quad\times Q^{(n-2)/2}_{-1/2-i\nu}\left(\cosh(r/R)\right), \end{aligned} \tag{3.5}$$

where $\epsilon = 0$ $(1/2)$ for n odd (even) and $\nu = \sqrt{2mR^2E}/\hbar$. The propagator is given by

$$\begin{aligned} K(r,T) &= \left(\frac{m}{2\pi i\hbar T}\right)^{1/2}\left(\frac{-1}{2\pi R\sinh(r/R)}\frac{d}{dr}\right)^{(n-1)/2} \\ &\quad\times\exp\{(i/\hbar)S_{cl}\}, \qquad \textit{for } n \textit{ odd}, \\ K(r,T) &= \sqrt{2}R\left(\frac{m}{2\pi i\hbar T}\right)^{3/2}\left(\frac{-1}{2\pi R\sinh(r/R)}\frac{d}{dr}\right)^{(n-2)/2} \\ &\quad\times\int_{r/R}^{\infty}\frac{t\exp\{imR^2t^2/2\hbar T\}}{\sqrt{\cosh t - \cosh(r/R)}}dt, \;\; \textit{for } n \textit{ even}, \end{aligned} \tag{3.6}$$

where $S_{cl} = mr^2/2T$ is the classical action. For $n = 2$ and 3 above results have already been obtained by Gutzwiller [7]. For $n = 1$ and $n = 3$ the semiclassical approximation is exact. Taking the flat space limit $R \to \infty$ the propagator reduces to the standard result. Finally we would like to mention that in the calculation of (3.5) it is crucial to consider the usual regularization $E \to E + i\delta$, $\delta > 0$, explicitly in order to obtain the correct result. This point has not been taken carefully in a recent calculation by Grosche and Steiner [8]. Indeed they have implicitly used the regularization $E - i\delta$ leading to an unphysical result.

3.2 The Klein-Gordon Propagator

As a second example we have chosen the relativistic propagator of a spinless particle in n space dimensions. The short-time propagator for this problem has already been given in Section 2. Here the transformation group is the n-dimensional Euclidean group being the semi-direct product of translations and rotations, $G = T^n \otimes SO(n)$. Taking $\mathbf{a}$ to be the origin, the stationary group is $H = SO(n)$. The corresponding zonal spherical functions $\mathcal{D}^k_{00}(g) = \Gamma(n/2)\,(2/kr)^{(n-2)/2}\,J_{(n-2)/2}(kr)$ depend only on the distance $r = |\mathbf{x}|$ from the origin, $\mathbf{x} = \mathcal{D}^{(n)}(g)\mathbf{a}$ [9]. The Fourier coefficient is found to be $f_k(\varepsilon) = \exp\left\{-(i/\hbar)\varepsilon c\sqrt{m^2c^2 + \hbar^2k^2}\right\}$ and $d_k = k^{n-1}/[2^{n-1}\pi^{n/2}\Gamma(n/2)]$. Note that the Lebesque measure in R^n is identical with the measure on $G/H \simeq T^n$ and therefore $|\mathcal{M}| = 1$. We immediately read off the energy spectrum $E_k = c(m^2c^2 + \hbar^2k^2)^{1/2}$. As $f_k(\varepsilon)$ is an exponential with exponent linear in ε the short-time propagator and the finite-time propagator are of the same form ($r = |\mathbf{x}_f - \mathbf{x}_i|$):

$$K(r,T) = 2icT\left(\frac{m\gamma}{2\pi i\hbar T}\right)^{(n+1)/2} K_{(n+1)/2}(imc^2T/\hbar\gamma). \tag{3.7}$$

In the above we have set $\gamma = [1 - r^2/c^2T^2]^{-1/2} = [1 - v^2/c^2]^{-1/2}$. Using the asymptotic expansion for small $\hbar$ we obtain the interesting representation

$$K(r,T) = \gamma\,(m\gamma/2\pi i\hbar T)^{n/2}\exp\{(i/\hbar)S_{cl}\} \times {}_2F_0(-n/2,(n+2)/2; i\hbar\gamma/2mc^2T), \tag{3.8}$$

where $S_{cl} = -mc^2T/\gamma$ is the classical action. For the unphysical values $n = -2$ and 0 the hypergeometric function in (3.8) becomes unity and the semiclassical approximation ($\hbar \to 0$) is found to be exact. Formula (3.8) explicitly demonstrates the equivalence of the classical ($\hbar \to 0$), large-time ($T \to \infty$), and non-relativistic limit ($c \to \infty$).

4 DISCUSSION AND OUTLOOK

Spacetime symmetry is certainly an important point to be considered in quantum theories. In this talk we have presented an alternative way for the path integration on symmetric spaces. The symmetry has been incorporated into the formalism and found to be very useful for the expilcit path integration. However, this technique is not limited to spacetime symmetries but may also be applied to problems having a dynamical symmetry (e.g. $SU(2)$ or $SU(1,1)$) [3,10]. Even to quantum field theories this method is applicable. For example, in lattice gauge theories the path integral is performed over the gauge and matter fields. They may also be changed into group integrals according to the above treatment.

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